BINOMIAL APPROACH FOR LOOKBACK OPTIONS

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While very respectful of the importance, usefulness, relative simplicity and popularity of the Binomial CCR Method and its authors, I must point out from a mathematical point of view, it is only one of countless numerical methods of solving partial differential equations. In addition, it is not the best one, according to Ioffe and Ioffe (1998). In particular, the Binomial CCR Method of solving Cauchy problem (the problem with the given initial conditions) or the problem with the boundary and initial conditions for parabolic equation of the second order, the specific (oblique) explicit finite difference scheme is used. Therefore, to use the Binomial Method we need to have both parabolic partial differential equation of the second order and initial and boundary conditions.

First, let's see how the mentioned equation and initial conditions appear in a Black-Sholes model while calculating premium of options, where pay-offs are defined only by stock price at expiration.

In this model the increase of stock price S(t) is defined by the equation:

$$\begin{aligned} &dS(t) = S(t)rdt + S(t)\sigma dW(t) \\ &0 \le t \le T \end{aligned} \quad \textbf{(1)}$$

Pay-off at expiration T is equal to:

$$C = e^{-rT}\phi(S(t)) \quad \textbf{(2)}$$

Function φ() is supposed to be given and defined by the kind of option. As it will be showed below, since we need to use the numerical method, this function has to satisfy some specific smooth conditions.

Let's make the following transformation of variable S(t):

$$x(t) = \frac{1}{\sigma} Ln(\frac{S(t)}{S(0)}) \quad \textbf{(3)}$$

It follows from (1) and (3) that increase of x(t) can be expressed by equation:

$$\begin{aligned} &dx(t) = \gamma dt + dW(t) \\ &\gamma = \frac{r - 0.5\sigma^2}{\sigma} \end{aligned} \quad \textbf{(4)}$$

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Pay-off at expiration T equals:

$$C(x) = e^{-rT}\phi(S(0)e^{\sigma x}) \tag{5}$$

It is obvious that process x(t) is Gauss, Markov's stochastic process depending only on the one variable γ. To avoid the arbitrage, option price has to be equal to the average of distribution of x(t), defined by equation (5). Without limitation, let's omit the discount in the following formulas.

Then option premium Pr can be expressed as integral:

$$\Pr = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x-\gamma t)^2} C(x)dx \tag{6}$$

Let's introduce a new variable z and consider the following function Ψ(z,t).

$$\Psi(z,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(z-x+\gamma t)^2} C(x)dx \tag{7}$$

Obviously, function Ψ(z,t) satisfies the following parabolic differential equation:

$$\frac{\partial\Psi(z,t)}{\partial t} = \gamma\frac{\partial\Psi(z,t)}{\partial z} + \frac{1}{2}\frac{\partial^2\Psi(z,t)}{\partial z^2} \tag{8}$$

Because of well-known property of density function of Gauss distribution, it is Dirac' Delta function at t=0, we have

$$\Psi(z,t=0)=C(z) \tag{9}$$

At z=0 function Ψ(z,t) is equal to the option premium. Therefore, any option with pay-off defined only by the stock price at the expiration moment satisfies the parabolic differential equation (8). Hence, to calculate the premium or to calculate integral (6) we can use any corresponding numerical method, in particular, the Binomial method.

For a path-dependent option, in particular a lookback option, it is much more difficult to find both the corresponding differential equation and initial conditions. Because we know explicit analytical solutions for calculating lookback options premium, inferences of which are based on the known formulas for density of distribution of maximum and minimum of given stochastic process, we can analyze the correctness of premium calculating in the Binomial model.

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We can write the known formula for calculating premium of floating strike call option as the following:

$$
\begin{aligned}
&\Pr CallFl=\int_{-\infty}^{\infty}\int_{-\infty}^{0}(S(0)e^{\sigma z}-S(0)e^{\sigma y})\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{2\pi t}}e^{2\gamma y}e^{-\frac{1}{2t}(z-\gamma t-2y)^{2}}\right)\chi(y,z)dzdy\\
&\chi(y,z)=1.....if....y<0..and..z>y \qquad\qquad \textbf{(10)}\\
&\chi(y,z)=0.....if....y<0..and..z<y
\end{aligned}
$$

Let's write the formula (10) as the average of function of Gauss variable, or as in (7).

$$
\begin{aligned}
&\Pr CallFl=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2t}(z-\gamma t)^{2}}\Theta(z,t)dz \qquad\qquad \textbf{(11)}\\
&\Theta(z,t)=\sqrt{2\pi t}\,e^{\frac{1}{2t}(z-\gamma t)^{2}}\int_{-\infty}^{0}(S(0)e^{\sigma z}-S(0)e^{\sigma y})\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{2\pi t}}e^{2\gamma y}e^{-\frac{1}{2t}(z-\gamma t-2y)^{2}}\right)\chi(y,z)dy
\end{aligned}
$$

It follows from formula (11) that pay-off of path-dependent options depends not only on stock price but also on the period of time from today to expiration. Hence, if we introduce function $\Psi 1(z,t)$ analogically to (7):

$$
\Psi 1(z,t)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2t}(z-x+\gamma t)^{2}}\Theta(x,t)dx \qquad \textbf{(12)}
$$

Then function $\Psi 1(z,t)$ will satisfy the equation:

$$
\frac{\partial\Psi(z,t)}{\partial t}=\gamma\frac{\partial\Psi(z,t)}{\partial z}+\frac{1}{2}\frac{\partial^{2}\Psi(z,t)}{\partial z^{2}}+\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2t}(z-x+\gamma t)^{2}}\frac{\partial\Theta(x,t)}{\partial t}dx \qquad \textbf{(13)}
$$

Therefore, when we use the Binomial Method for premium calculating, there is a calculation error that is connected with dependence of pay-off on time. If the function is given, as it is in the considering example when we know the exact analytical solution, then calculating the integral in the right part of (13) we can take into account this error.

For options with little time before expiration we can simplify the calculation, having:

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2t}(z-x+\gamma t)^{2}}\frac{\partial\Theta(x,t)}{\partial t}dx\cong\frac{\partial\Theta(z+\gamma t,t)}{\partial t} \qquad\qquad \textbf{(14)}\\
&\frac{\partial\Theta(z+\gamma t,t)}{\partial t}\cong\frac{\Theta(\gamma t,t+\Delta t)-\Theta(\gamma t,t)}{\Delta t}
\end{aligned}
$$

We can use the same approach for options with Fixed Strike K, and function Θ1(z,t) for Call can be described analogically to (11) as:

(15)

$$\Pr CallStr = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(z-\gamma t)^2} \Theta 1(z,t)dz$$

$$\Theta 1(z,t) = \sqrt{2\pi t}\, e^{\frac{1}{2t}(z-\gamma t)^2} \int_{yk}^{\infty} (S(0)e^{\sigma y} - K)\frac{\partial}{\partial y}(-\frac{1}{\sqrt{2\pi t}} e^{2\gamma y} e^{-\frac{1}{2t}(z-\gamma t-2y)^2})\chi 1(y,z)dy$$

$$\chi 1(y,z) = 1....if....y > yk.....and....z < y$$

$$\chi 1(y,z) = 0....if....y > yk.....and....z > y$$

$$yk = \frac{1}{\sigma} Ln(\frac{K}{S(0)})$$

It should be mentioned that the integrals in formulas (11) and (15) are explicitly calculated by means of using Gauss distribution function. Basing on the stated above method for parabolic nonhomogeneous equation (13) with initial conditions (11) or (15) and using finite difference scheme corresponding to the Binomial Method, premiums for the following options were calculated, referencing Cheuk and Vorst (1997), for:

1. European floating strike lookback call(S(0)=100,rd=0.04,rf=0.07, T=0.5)
2. European fixed strike lookback call(S(0)=100, K=100,rd=0.04,rf=0.07, T=0.5)

In both cases the decision has converged if we have nstep=50.
All results are represented in Table 1. (nstep=50)

Tab. 1

	Option price		
	σ=0.1	σ=0.2	σ=0.3
Binomial float	4.6810	9.7945	14.6919
Analitic float	4.6799	9.7915	14.6872
Binomial fixed	6.7066	13.8602	21.3750
Analitic fixed	4.9231	10.7644	16.8762

As can be seen in the Table 1, the errors of definition of premium for European floating strike lookback call are insignificant; meanwhile, those for European fixed strike lookback call reach 40%. There are three kinds of errors in calculating premium of path-dependent options with finite difference method:

1. error in definition of pay-off function
2. error brought about by dependence of pay-off function on time
3. error of numerical method brought about by representation of solution of differential equation as the solution of finite difference equations. Error in arithmetical calculations also belongs to this kind of errors.

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In our example, there are no errors of the first kind, because we use the known analytical equation for definition of pay-off function of lookback options. Increasing the number of steps in the difference scheme, we can make the errors of the third kind very small. In our example, as there is conversion on the first 50 steps, we can suppose that there are almost no errors of the third kind. Hence, the main source of errors in our approximation is the dependence of pay-off function on time.

If we know the exact analytical solution, as in our example, we can diminish this error by means of using more complex approximation of dependence of pay-off function on time. In general cases, not using the known analytical solution, to realize the Binomial Method it is necessary to define pay-off function at the expiration moment. It is supposed by Cheuk and Vorst (1997) that for European floating strike lookback call option this formula is equal to (see formula <7>) $S(t_N)(1-u^{-k})$, where $u=e^{\sigma\sqrt{delt}}$, N=nstep, k-the number of steps up.

In classic binomial node's values of stock price are equal to:

$$S(i,k) = S(0)u^{(2k-i)}$$
$$0 \le i \le nstep, 0 \le k \le i \qquad \textbf{(16)}$$

Because of our assumption we can consider that pay-off function in nodes $F_0(k)$ is equal to:

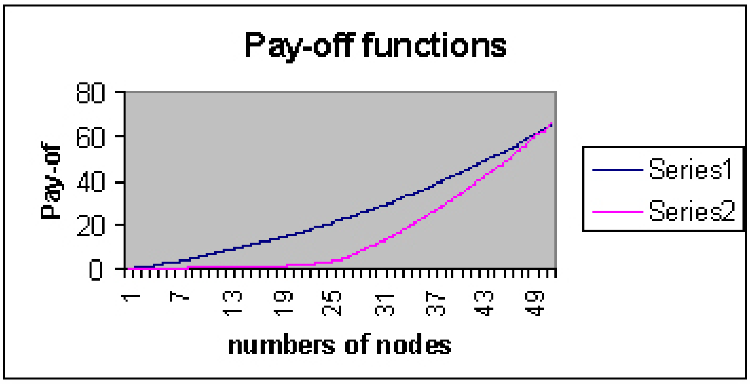
$$F_0(k) = S(0)(u^{(2k-nstep)} - u^{(k-nstep)})$$
$$0 \le k \le nstep \qquad \textbf{(17)}$$

It follows from formula (3) and (16) that the corresponding values of variable z(k) are equal to:

$$z(k) = (2k - nstep)\sqrt{delt} \qquad \textbf{(18)}$$

Using the formulas (11) and (18) we can calculate the real value of pay-off function in node $F_1(k)$. You can see the results of calculations on the **Figure 1**. Series 1 corresponds to the pay-off function in knot $F_0(k)$ from Cheuk and Vorst (1997); Series 2 corresponds to the real values that can be calculated if we know option price.

Fig. 1



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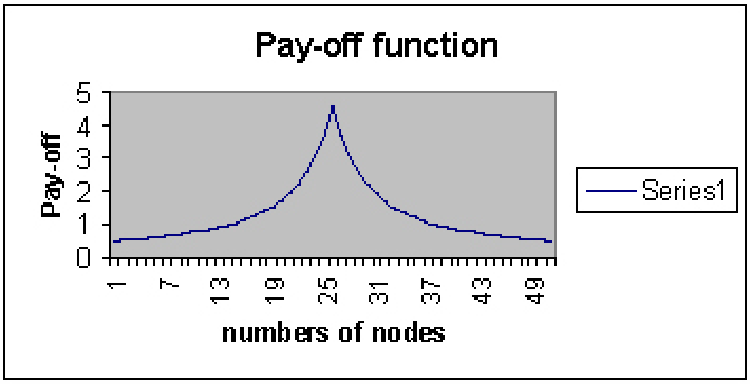
As it can be seen in Fig.1, the supposed function is significantly different from the real function. To correct this difference in initial values in the theory demonstrated by Cheuk and Vorst (1997) it is used a non-classical scheme of Binomial Method - formulas (10) and (11)) used by Cheuk and Vorst (1997). In the classical scheme of Binomial Method, the price of the option in k-knot on i-step is calculated with the option price in k+1 knot and k knot on I+1 step. Cheuk and Vorst (1997 suppose to calculate option price in k knot on i step with the option price in k+1 and k-1 knot on i+1 step. If we use this schema for calculating the price of European call with parameters S(0)=100,Strike=100,rd=0.04,rf=0.07, σ=0.1, T=0.5, then after 50 steps we will have the value 2.2427 while the real value, calculated from the Black-Sholes formula or calculated with classical Binomial Method with nstep=50, is 2.0758. But even with these changes of Binomial Method we can rely on the results only if we simulate the significant number of steps (~5000).

Cheuk and Vorst (1997) also contend that for European fixed strike lookback call pay-off function is equal to zero. The real value of this function that is calculated with formulas (15) and (18) is shown on **Figure 2**.

Fig. 2



To have a reliable result in this case, Cheuk and Vorst (1997) suppose to solve a non-homogeneous equation with the right part.

Conclusions.
1. To apply Binomial Method for the calculating of path-dependent options it is initially necessary to solve equally difficult problem, the problem of definition of pay-off function at the expiration moment.
2. It is necessary to take into account in Binomial schema that pay-off function depends on time.
3. If there is no known analytical solution, or benchmark, the results of Binomial Method are doubtful.

References:
1. G.Ioffe and M.Ioffe. EGAR Technology Focus White paper: Application of finite Difference Method for Pricing Barrier Option 1998
2. Cheuk T.H.P. and Vorst A.C.P. Currency lookback options and observation frequency: a binomial approach. Journal of International Money and Finance, Vol 16,No. 2, pp 173-187, 1997

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