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European and American options with a single payment of dividends

(About formula Roll, Geske & Whaley)

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Abstract

The article provides a derivation of formulas for pricing European and American Call options with a single payment of dividends. For a European option derivation is based on the distribution of the stock price at the expiration time. For the American option is derived and numerically solved an equation for the option price as a function of the stock price and time until expiration. Numerical results are compared to calculations derived from calculations based on the formulas currently used methods. It turns out that the American option results to differ essentially from the method of Roll, Geske & Whaley.

Interestingly, this particular case of options, when dividends are paid only once, is analyzed in detail in the option theory and, in particular, online http://www.global-derivatives.com in the section on American options. This section provides a method for calculating the American Call options for stocks with a single payment of the dividend, developed by Roll, Geske & Whaley. Apparently, this method is implemented in Matlab financial toolkit function optstockbyrgw. Let us first consider how to calculate price for the European Call with single dividend payment. Under standard assumptions, when the change in the stock price is described by a diffusion process with constant parameters, stock's price at expiration time could be described by the formula:

$$S(t_1 + t_2) = (S_0 e^x - D)e^y$$

(1)

Where

t_1 - is the time to pay a dividend;

t_2 - is the time from payment of a dividend to expiration;

S_0 - share price at the time of settlement;

D - dividend amount;

x - Gauss random variable with mean and standard deviation:

$$\text{«}_x = (r_d - 0.5\pm^2)t_1$$

$$\pm_x = \pm\sqrt{t_1}$$

(2)

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y - Gauss random variable with mean and standard deviation:

$$\text{«}_y = (r_d - 0.5\pm^2)t_2$$
$$\pm_y = \pm\sqrt{t_2}$$

(3)

In (2) and (3)
r_d - free interest rate;
σ - volatility.

Random variables x and y are independent. Density distribution of these variables equals:

$$p_x(u) = \frac{1}{\sqrt{2\text{®}}\pm_x} e^{-\frac{1}{2\pm_x^2}(u-\text{«}_x)^2}$$
$$p_y(u) = \frac{1}{\sqrt{2\text{®}}\pm_y} e^{-\frac{1}{2\pm_y^2}(u-\text{«}_y)^2}$$

(4)

Total price of the Call option with a strike K obviously:

$$C = e^{-r_d(t_1+t_2)} \iint\limits_{(S_0e^x - D)e^y > k} ((S_0e^x - D)e^y - K)p_x(x)p_y(y)dxdy$$

(5)

Equation (5) can be written as:

$$C = e^{-r_d(t_1+t_2)} \int\limits_{-\infty}^{\infty} p_y(y)dy \int\limits_{Ln(\frac{k+De^y}{S_0})}^{\infty} (S_0e^{x+y} - De^y - K)p_x(x)dx$$

(6)

Using the cumulative normal distribution function (Laplace function) Lp (x):

$$Lp(x) = \frac{1}{\sqrt{2\text{®}}} \int\limits_{-\infty}^{x} e^{-\frac{1}{2}u^2} du$$

(7)

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Present (6) as:

$$C = e^{-r_d(t_1+t_2)} \int_{-\infty}^{\infty} p(y)(S_0 e^{y+\mu_x+0.5\sigma_x^2} Lp(-\frac{xk-\mu_x-\sigma_x^2}{\sigma_x}) - (De^y + K)Lp(-\frac{xk-\mu_x}{\sigma_x}))dy(y)$$

$$xk = Ln(\frac{Ke^{-y}+D}{S_0})$$

(8)

The integral in (8) can be calculated, for example, using Hermite polynomials. We set

$$z = \frac{y-\mu_y}{\sqrt{2}\sigma_y}$$

(9)

Substituting (9) into (8), we obtain

$$C = e^{-r_d(t_1+t_2)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-u^2} (S_0 e^{y+\mu_x+0.5\sigma_x^2} Lp(-\frac{xk-\mu_x-\sigma_x^2}{\sigma_x}) - (De^y + k)Lp(-\frac{xk-\mu_x}{\sigma_x}))du$$

$$xk = Ln(\frac{Ke^{-y}+D}{S_0})$$

$$y = \sqrt{2}\sigma_y u + \mu_y$$

(10)

To calculate the integral in (10) we apply Gauss numerical integration using Hermite polynomials. According to this formula:

$$\int_{-\infty}^{\infty} e^{-u^2} \varphi(u)du = \sum_{i=1}^{n} w(i)\varphi(x(i))$$

(11)

Where

x (i) - the roots of n-order Hermite polynomial;
w (i) - weights.

Formulas (10) and (11) allow one to accurately calculate the price of the European option with one dividend. Obviously, in the case where the parameter D is 0 (10) and (11) should show the value of the option price, calculated according to the formula Black-Scholes. Implemented in Excel for an option with the parameters S_0 = 20, K = 22, rd = 7%, σ = 20%, D = 0, t1 = 0.252055 (92 days), t2 = 0.419178 (153 days), these formulas give the value of the option price 0.889396. The Black-Scholes formula gives 0.889397.

In practice, European option pricing with dividends generally carried out by the Black-Scholes option or by typing there r_f-continuous dividend rate or by reducing the S_0. And in both cases, the reference is to the average distribution for the share price at the expiration time MS, obviously equal to:

$$MS = S_0 e^{r_d(t_1+t_2)} - De^{r_d t_2}$$

(12)

In the first case, the parameter r_f is the solution of equation:

$$S_0 e^{(r_d - r_f)(t_1+t_2)} = S_0 e^{r_d(t_1+t_2)} - De^{r_d t_2}$$

(13)

That is:

$$r_f = -\frac{1}{t_1+t_2} Ln(1 - \frac{D}{S_0} e^{-r_d t_1})$$

(14)

In the second case, the parameter S_{00} is the solution of equation:

$$S_{00} e^{r_d(t_1+t_2)} = S_0 e^{r_d(t_1+t_2)} - De^{r_d t_2}$$

(15)

That is:

$$S_{00} = S_0 - De^{-r_d t_1}$$

(16)

Table 1 shows the values of the calculated option prices with the above parameters with 4 values dividends specified in column D, for the four methods of calculation: in columns Pr1, Pr2 adjusted continuous dividend rate and stock prices, respectively, in the column Pr3 value obtained using the calculator on the website www.ivolatility.com, column Pr4 value obtained using (10) and (11).

Table 1.

D	Pr1	Pr2	Pr3	Pr4
2	0.299270577	0.299271	0.2979	0.333408
4	0.063437521	0.063438	0.063	0.095693
6	0.006842135	0.006842	0.0068	0.020605
8	0.000270507	0.000271	0.0003	0.003314

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As shown in Table 1, the traditional way to calculate European option with a dividend yield values that are different from the values prescribed in the model.
However, the differences are very small.

We next consider the American option. According to the definition, the buyer of the option can exercise it at any time from purchase to expiration and for this right to pay in general, some amount cash. It is generally accepted that early expiration occurs if and only if the price of the option does not exceed the difference between the stock price and the strike price. Price of American Call PrC (S, t) as a function of the stock price S and time to expiration t, satisfies the following equation:

$$\Pr C(S,t) = Max(S-K, \int_0^{\infty} p(S,Q,\Delta t)\Pr C(Q,t-\Delta t)dQ)$$

(17)

Where
p (S, Q, Δt) - probability density of price changes from the value S to the value of Q in the time Δt. According to the log-normal model of stock price changes, we can write:

$$Q = Se^{\Delta y}$$
(18)

Where
Δy - Gauss random variable with mean and standard deviation:

$$«_{\Delta y} = (r_d - 0.5±^{2})\Delta t$$
$$±_{\Delta y} = ±\sqrt{\Delta t}$$

(19)

The random variables S and Δy - independent. The probability density of price changes from the value S to the value of Q in the time Δt can be calculated as follows. The joint cumulative distribution function of S and Q is:

$$F(u,v) = \Pr ob(S < u, Se^{\Delta y} < v) = \Pr ob(S < u, \Delta y < Ln(\frac{v}{S})) = \int_0^{u} p_S(x)dx \int_{-\infty}^{Ln(\frac{v}{x})} p_{\Delta y}(y)dy$$

(20)

Where
P_S (x) - the density of the random variable S;

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$P_{\Delta y}(x)$ - the density of the random variable Δy.

The density distributions of S and Q is:

$$p_{S\Delta y}(u,v) = \frac{d^2F(u,v)}{dudv} = p_s(u)\,p_{\Delta y}(Ln(\frac{v}{u}))\frac{1}{v}$$

(21)

From (21) that the probability density of price changes from the value S to the value of Q in the time Δt, which is the conditional probability density is:

$$p(u,v,\Delta t) = p(Q\,|\,S,\Delta t) = \frac{p_{S\Delta y}(u,v)}{p_s(u)} = p_{\Delta y}(Ln(\frac{v}{u}))\frac{1}{v}$$

(22)

In accordance with (19)

$$p_{\Delta y}(y) = \frac{1}{\sqrt{2\pi}\,\sigma_{\Delta y}} e^{-\frac{1}{2\sigma_{\Delta y}^2}(y-\mu_{\Delta y})^2}$$

(23)

From (22), the probability density of the stock price changes from the value S to the value of Q in the time Δt is independent of the distribution of the stock price S, that is the same before and after the payment of a dividend.

To calculate the price of an American Call option with one dividend is necessary to solve for the unknown function PrC (S, t), equation (17).The initial condition for the solution is obvious:

$$\Pr C(S,0) = \max(S-K,0)$$

(24)

On the time interval from the time of payment of a dividend until expiration this function is the Black-Scholes formula for a European Call option, since it is known that in the absence of dividends, the value of American and European Call is equal. Show it, that is, we prove the inequality with the Black-Scholes formula:

$$F = SLp(u) - Ke^{-rdt} Lp(u1) - (S - K) \geq 0$$

$$u = \frac{Ln(\frac{S}{K}) + (rd - 0.5\sigma^2)t}{\sigma\sqrt{t}}$$

$$u1 = u - \sigma\sqrt{t}$$

$$Lp(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{1}{2}x^2} dx$$

(25)

Since the derivative of the option price by rd equals

$$\frac{\partial C}{\partial rd} = Kte^{-rdt} Lp(u1)$$

(26)

it is always positive. Therefore, the minimum value of the function is reached at rd = 0. Therefore put on rd = 0, and in this case we will prove inequality.

We set

$$x = \frac{S}{K}$$

$$y = \pm\sqrt{t}$$

$$F = K(x(Lp(u) - 1) + 1 - Lp(u1))$$

$$u = \frac{\ln(x) + \frac{1}{2}y^2}{y}$$

$$u1 = \frac{\ln(x) - \frac{1}{2}y^2}{y}$$

(27)

It is easy to show that

$$\frac{\partial F}{\partial x} = Lp(u) - 1 \le 0$$
$$\frac{\partial F}{\partial y} = \frac{x}{\sqrt{2\circledR}} e^{-\frac{1}{2}u^2} \ge 0$$

(28)

Obviously, we are only interested in the values of S>K (x>1, ln(x)>0).

Because of (28) that the derivative of F with respect to y for any x is non-negative, the function F has a minimum value at y = 0, and since Lp(∞)=1, the minimum value is 0, which proves the inequality.
Thus, in the time after the payment of the dividend option price is:

$$\Pr C(S,t_2) = SLp(u) - Ke^{-r_d t_2} Lp(u - \pm\sqrt{t_2})$$
$$u = \frac{Ln(\frac{S}{K}) + (r_d - 0.5\pm^2)t_2}{\pm\sqrt{t_2}}$$

(29)

Because after the payment of a dividend stock price is reduced by the amount of dividends paid prior to payment of the option price is:

$$\Pr C\_(S,t_2) = \Pr C(S+D,t_2)$$
(30)

For the calculation of the option price at the initial time, separated from him for a time prior to the payment of the dividend t_1, we divide the whole interval into n segments and for the transition from one to the other, we use formula (17). That is

$$\Pr C(S,t(k+1)) = Max(S-K, \int_0^\infty p(S,Q,\Delta t) \Pr C(Q,t(k))dQ)$$
$$t(1) = t_1$$
$$t(n+1) = 0$$
$$t(k) = \frac{t_1}{n}(n+1-k)$$

(31)

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In accordance with formulas (22) and (23)

$$p(S,Q,\Delta t) = p_{\Delta y}(Ln(\frac{Q}{S}))\frac{1}{Q} = \frac{1}{\sqrt{2\pi}\sigma_{\Delta y}Q}e^{-\frac{1}{2\sigma^2_{\Delta y}}(Ln(\frac{Q}{S})-\mu_{\Delta y})^2}$$

(32)

We set

$$S = S_0 e^x$$
$$Q = S_0 e^y$$

(33)

With (32) and (33) (31) can be written as

$$\Pr C(x,t_{k+1}) = Max(S_0e^x - K, \int_{-\infty}^{\infty}\Pr C(y,t_k)\frac{1}{\sqrt{2\pi}\sigma_{\Delta y}}e^{-\frac{1}{2\sigma^2_{\Delta y}}(y-x-\mu_{\Delta y})^2}dy)$$

(34)

To calculate the integral in (34) we again use the numerical method of Gauss, that is by (11), first converting the variable of integration. That is

$$\Pr C(x,t_{k+1}) = Max(S_0e^x - K, \int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-u^2}\Pr C(\sqrt{2}\sigma_{\Delta y}u,t_k)e^{-\frac{(x+\mu_{\Delta y})}{2\sigma^2}(-2\sqrt{2}\sigma_{\Delta y}u_i+(x+\mu_{\Delta y})\ )}du)$$

$$u = \frac{y}{\sqrt{2}\sigma_{\Delta y}}$$

(35)

Thus, the

$$\Pr C(x_j,t_{k+1}) = Max(S_0e^{x_j} - K, \sum_{i=1}^{n}\frac{1}{\sqrt{\pi}}w_i\Pr C(\sqrt{2}\sigma_{\Delta y}u_i,t_k)e^{-\frac{(x_j+\mu_{\Delta y})}{2\sigma^2_{\Delta y}}(-2\sqrt{2}u_i\sigma_{\Delta y}+(x_j+\mu_{\Delta y}))})$$

(36)

Table 2 shows the values of the calculated price of the American option with the above parameters with the four values of the dividends specified in column D, the four ways of computing: the column AmMy value obtained using (30) and (36), the column AmIvol value obtained from Calculator online www.ivolatility.com, column AmBin binomial, adjusted continuous dividend rate, the column MatLab value obtained by MatLab function optstockbyrgw.

Table 2.

D	AmMy	AmIvol	AmBin	MatLab
2	0.3811	0.3752	0.299271	0.2237
4	0.2631	0.2711	0.115645	0.1544
6	0.2462	0.2583	0.032329	0.1068
8	0.2451	0.2577	0.007187	0.0664

As shown in Table 2, the calculated by our formula values fundamentally differ from the values obtained by method Roll, Geske & Whaley (MatLab function optstockbyrgw).