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About mathematically correct definition of American option premium and using Monte-Carlo method for its pricing

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Abstract.

For the mathematically correct definition of the American option premium you must specify a sequence of points in time at which the conditions of the early expiration tests and the assumption that between two successive moments of testing option is a European. There is no reason to believe that using Monte-Carlo method for American option pricing is in something superior to known binomial.

What is mathematically correct definition of American option premium? By mathematically correct definition we understand the definition that has two properties. First, by virtue of this definition, we can construct a numerical algorithm to calculate the premium. Second, the definition should allow one and only one value. Below without loss of generality, we will consider the option Call. Also, assume that r (rate interest) = 0 and that there are no dividends. Recall the definition of American Call. American Call is the option which premium at the time of expiration is equal Max (S-K, 0), (S - stock price, K - strike price) but it can be exercised at any time t from the time t_0 of writing to the time t_{exp} of expiration with premium S-K, if at this time the premium will be less than S-K. Assuming that the price of the option at a given time is a function of the known values the stock price and time to expiration, i.e.

$$C = \phi(S,t)$$

(1)

We will measure time from expiration. Then:

$$\phi(S,0) = Max(S-K,0)$$

(2)

Only one condition (2) is not sufficient to determine the American option premium, since there is no value needed for comparison with the difference between the spot and the strike that is not possible to verify the condition of early expiration. The most logical is to set the sequence of moments of time in which to conduct testing and the assumption that between two successive moments of testing option is a European. Thus, if the function $\phi(S,t)$ known at time t (i), then at time t (i + 1)

$$\phi(S(t(i+1)),t(i+1)) = Max(\phi 1(S(t(i+1)),t(i+1)),S(t(i+1))-K)$$

(3)

Where

$\phi 1(S(t(i+1)),t(i+1))$ - European option premium at time t (i + 1), if at time t (i) it equals to $\phi(S(t(i)),t(i))$.

In general this value can be calculated by the formula:

$$\phi 1(S(t(i+1)),t(i+1)) = \int_0^\infty p(t(i+1)-t(i),S(t(i+1)),S)\phi(S,t(i))dS$$

(4)

Where

$p(t(i+1)-t(i),S(t(i+1)),S)$ - probability density of transition of the stock price from the value S (t (i + 1)) to S when time changes from t (i + 1) to t (i)

To write the formula (4), we used to be widely used, for example, in Cox, Rubinstein "Options market" book property, which consists in the fact that at any given time with a certain stock price premium is equal to the expectation of the distribution of the premium at a subsequent time, that is, sum of the products of premiums at a subsequent time in the stock price by the transition probability from a known value to any possible value.

Consider the hypothetical singular case when we know at any given time from t_0 to t_{exp} stock price, i.e., the function $S_{(t)}$, $t_0 \le t \le t_{exp}$.

Obviously, in this case the probability density equal to Dirac delta function, i.e.

$$p(t(i+1)-t(i),S(t(i+1)),S) = \delta(S-S(t(i+1)))$$

(5)

In this case, since the process of changing stock price is considered deterministic, there is only one possible value in the subsequent point and the transition probability is equal to one.

Let $t_1 < t_2 < t_3$ ... .. $< t_n$ - times in the interval (t_0, t_{exp}). Since premium at expiration time is defined, for a known time it can be determined using a recursion at any moment, and in particular t_0, according to the formulas:

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$$\begin{aligned}
&\Pr(i+1) = \max(\Pr(i), S(t(i+1)) - K) \\
&i = 0,1,\ldots.n \\
&t(0) = t\exp \\
&t(n+1) = t0 \\
&t(i) = ti
\end{aligned}$$

(6)

Thus, for any finite set of time points in the interval (t_0, t_{exp}) premium is uniquely determined. From the formula (6), it obviously follows that:

$$\Pr(t0) = \max_{i=0,1,..n+1}(S(t(i)) - K)$$

(7)

What happens if the number of points of the partition of the interval (t_0, t_{exp}) tends to ∞, and the maximum length of a segment of the partition tends to zero? It is obvious that if the function $S_{(t)}$ is continuous, then a limit exists and premium equals max ($S_{(t)}$ - K). In general, the situation is more complex, that is, the limit may not exist. For example, if $S_{(t)}$ is equal to a known Riemann function (0 for rational numbers and 1 for irrational) limit, obviously, does not exist and the definition of the American option premium becomes incorrect. Despite the rather exotic example of a function, this example demonstrates the need to set a series of time points at which tests the possibility of early expiration.

We next consider the more realistic and widely used model for the process of change in stock price $S_{(t)}$ (Black-Sholes environment). Here stock price $S_{(t)}$ is described by the equation for the diffusion process:

$$\frac{dS(t)}{S(t)} = (r_d - r_f)dt + \sigma dW(t)$$

(8)

Where

r_d – rate interest

r_f – dividend interest

σ – volatility

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$W_{(t)}$ Wiener process.

In the theory of diffusion stochastic processes is proved that any function φ (S, t), has the form:

$$\varphi(S,t) = M_{S,t}(f(S(T),T)$$
$$t < T$$

(9)

Where

M () – mathematical expectation

satisfies the parabolic partial differential equation:

$$\frac{\partial\varphi}{\partial t} = -(r_d - r_f)S\frac{\partial\varphi}{\partial S} - \frac{1}{2}S^2\sigma^2\frac{\partial^2\varphi}{\partial^2 S^2}$$

(10)

with the initial conditions:

φ (S, t = T) = f (S, T)

(11)

If the time count from the T, which in this case is the expiration time, i.e. putting τ = T-t, then equations (10) and (11) can be written as:

$$\frac{\partial\varphi}{\partial\tau} = (r_d - r_f)S\frac{\partial\varphi}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2\varphi}{\partial^2 S^2}$$

φ(S,0)=f(S,0)

(12)

It is obvious that if a function f (S, 0) = max (S-K, 0) and solve the equation (11) with this initial condition, the resulting solution will famous Black-Sholes formulas to calculate the European option premium. With regard to the premium of the American option, that in spite of the work of Giovanni Barone-Adesi and Robert E / Whaley, «Efficient Analytic Approximation of American Option Values", The Journal of Finance, Vol. XL ||, No / 2, June

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1987, two simple considerations show that the premium of the American option does not satisfy this equation. The first consideration is that the function S-K provided r_d -$r_f \neq 0$, equation (12) does not satisfy. However, there is a range of parameters S and τ, where the premium American option is equal S-K. The second consideration is that, since the initial and boundary conditions (S $\to \infty$, S $\to$ 0) for European and American options are the same, the satisfaction of premium American options equation (12) would be contrary to the uniqueness of the solution of the problem.

Thus, for the mathematically correct definition of the American option premium, you must specify a sequence of points in time at which the conditions of the early expiration tests. In this case, the calculation of its premium is reduced to a recurrent solution of equation (12), the initial conditions for the next time period of time determined by integrating the equation in the previous step. Such a method used in a finite-difference scheme, or solutions by the binomial method.

Next, consider possibility to calculate an American option by Monte Carlo method. As is known, this method is the simulation of a large number of trajectories of stock price changes in accordance with the model described in equation (8), i.e. in accordance with the geometric random walk model. Sequence of stock prices can be calculated by the formulas:

$$S(i) = S(i-1)e^{N(\mu,\sigma t)}$$
$$S(0) = S_0$$
$$\mu = (rd - rf - 0.5\sigma^2)\delta t$$
$$\sigma t = \sigma\sqrt{\delta t}$$

(13)

Where

$N(\mu,\sigma t)$ - Gaussian random variable with a mathematical expectation μ and standard deviation σt. This value can be obtained from uniformly distributed in the interval [0, 1] variant Rnd () in different ways. For example, as shown in the book Forsythe, GE / Malcolm, MA / Moler, CB, Computer Methods for Mathematical Computations. Englewood Cliffs, New Jersey 07632. Prentice Hall, Inc., 1977. XI, 259 S, it has the following formulas:

$$
\begin{aligned}
&100: x = Rnd()\\
&x1 = 2x - 1\\
&x = Rnd()\\
&x2 = 2x - 1\\
&s = x1^2 + x2^2\\
&if..s > 1...then..Goto..100\\
&u1 = x1\sqrt{-\frac{2Ln(s)}{s}}\\
&u2 = x2\sqrt{-\frac{2Ln(s)}{s}}\\
&N(\mu, \sigma t) = \frac{\sigma t}{\sqrt{2}}(u1 + u2) + \mu
\end{aligned}
$$

(14)

In another method of generating a normal variant used the inverse function of the cumulative normal distribution function, i.e.

$$
\begin{aligned}
&N(\mu, \sigma t) = Lp^{-1}(Rnd(), \mu, \sigma t)\\
&Lp(y, \mu, \sigma t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{y-\mu}{\sigma t}} e^{-\frac{1}{2}u^2} du
\end{aligned}
$$

(15)

In most cases, when the option premium is uniquely determined for each of the random trajectory estimation of the possible premium for the whole set of possible trajectories calculated as the average premium for each trajectory premium. For example, for vanilla Call at rd = rf = 0 option price is estimated as:

$$
\Pr = \frac{1}{N}\sum_{i=1}^{N} \max(S(i, t\exp) - K, 0)
$$

(16)

Where

N - amount of the generated trajectories.

When calculating the premium of the American option for each individual path, as indicated above, premium is not defined. Therefore, this approach is a simple averaging does not work. Nevertheless, in the literature using the Monte Carlo method for calculating American options are well represented. The most known and one of the

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first is the work of *"Valuing American Options by Simulation:* A Simple Least-Squares Approach. Francis A. Longstaff. UCLA. Eduardo S. Schwartz. UCLA". In this paper, a first step for calculating is generation by Monte Carlo of set of trajectories describing changes in share prices. Further calculation is carried out recursively. At the expiration time option price is determined uniquely, that is, for example, a put option is equal to the difference between the strike price and the stock price, if the difference is positive and zero otherwise. The generated set of paths is a grid. On this grid value of an option is calculated recursively in time. To move from one point in time to an earlier for some small amount of nodes, in particular those where the option price is not equal to 0, it is assumed that the value of a European option for these nodes is equal to the discounted values of the option price in the previous nodes. In fact, for the calculation should use either the Black-Scholes formula, or similar to it, that is, to calculate the integral:

$$\Pr(i,S) = e^{-rd\delta t}\int_0^\infty \frac{1}{\sqrt{2\pi}\sigma t x} e^{-\frac{1}{2\sigma t^2}(Lnx - LnS - \mu)^2} \Pr(i+1,x)dx$$

(17)

The integral in (17) can be calculated in various ways, including using the Monte Carlo method. More appropriate is to use the method of its calculation using the Gauss- Hermite polynomials.

Obviously, assumption that has been done in the above-mentioned work is valid only in the case when the intervals between the points of the trajectories are small, number of steps is large, or when volatility is low. This substantially limits the method of calculation. It seems that it is unlikely to save method by using the method of least squares for approximation of value option price at the nodes of polynomials of low order. There is no reason to believe that this method of calculation is in something superior to known binomial.