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About M. Curran's method for Asian options

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Abstract

The article refers to the calculation of the cost of Asian option. In particular, we show that the realization of proposed M. Curran's method for Asian option [1] based on the fact that the Arithmetic Mean is always no less than the Geometric Mean can be greatly simplified. The article presents how it can be done.

Payoff at expiration T for Asian Call option equals:

$$C = Max(\frac{1}{n}\sum_{i=1}^{n} S(t_i) - K, 0)$$

(1)

Where

$S(t_i)$ – stock price at time t_i;
K – Strike price;
N – number of averaging.

In accordance with the Black-Sholes model price change stock S (t) is given by:

$$dS(t) = S(t)(r_d - r_f)dt + S(t)\sigma dW(t)$$
$$0 \le t \le T$$

(2)

Where

r_d = constant interest rate;
r_f = constant dividend rate;
σ = constant volatility.

That is payoff at expiration T equals:

$$\Pr C = Max(\frac{1}{n}\sum_{i=1}^{n} e^{x(i)} - K, 0)$$

(3)

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Where x (i) = Ln (S (i)), i = 1,2 ... n the values of the logarithms of the respective stock prices at averaging time t (i). In accordance with the Black-Sholes model the random variables x (1), x (2) ... .x (n) form a normal, Gaussian vector, with the vector of the expectation of m (i) and covariance matrix Covmat (i, j), calculated according to formulas:

$$m(i) = LnS_0 + (r_d - r_f - 0.5\sigma^2)t(i)$$
$$Covmat(i,j) = \sigma^2 Min(t(i), t(j))$$

(4)

Where

S_0 – initial stock price value.

Non arbitrage option price is the expectation of the random variable PrC from the formula (3).

$$\Pr Call = e^{-r_d T}\int_{-\infty}^{\infty}....\int_{-\infty}^{\infty} Max(\frac{1}{n}\sum_{i=1}^{n} e^{x(i)} - K, 0)\, p(x(1), x(2)...x(n), m, Covmat)dx(1)dx(2)...dx(n)$$

$$p(x(1), x(2)...x(n), m, Covmat) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{Det(Covmat)}} e^{-\frac{1}{2}(x-m)^T Covmat^{-1}(x-m)}$$

(5)

Method of calculating of non arbitrage price of the option proposed by Michael Curran, based on a simple fact: the Arithmetic Mean of all n values is always no less than the Geometric Mean of these same quantities. This fact in the following way can be used to calculate a price.

$$\Pr Call = \Pr Call1 + \Pr Call2$$

$$\Pr Call1 = e^{-r_d T}\int_{\frac{1}{n}\sum_{i=1}^{n} x(i) > LnK} .....\int (\frac{1}{n}\sum_{i=1}^{n} e^{x(i)} - K)\, p(x(1), x(2)...x(n), m, Covmat)dx(1)dx(2)...dx(n)$$

$$\Pr Call2 = e^{-r_d T}\int_{\frac{1}{n}\sum_{i=1}^{n} x(i) < LnK} .....\int Max(\frac{1}{n}\sum_{i=1}^{n} e^{x(i)} - K, 0)\, p(x(1), x(2)...x(n), m, Covmat)dx(1)dx(2)...dx(n)$$

$$G = \left(\prod_{i=1}^{n} e^{x(i)}\right)^{\frac{1}{n}}$$

(6)

Integral PrCall1 can be written as:

$$\Pr Call1 = e^{-r_d T}\frac{1}{n}\sum_{i=1}^{n}\int_{\frac{1}{n}\sum_{i=1}^{n} x(i) > LnK} .....\int (e^{x(i)} - K)\, p(x(1), x(2)...x(n), mCovmat)dx(1)dx(2)...dx(n)$$

(7)

How it can be calculated see [2].

To calculate PrCall2 believe that the Arithmetic Mean of the log. normal values itself has a log. normal distribution, i.e.

$$A = \frac{1}{n}\sum_{i=1}^{n} e^{x(i)} = e^{x}$$

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2\sigma_A^2}(x-m_x)^2}$$

(8)

This distribution unknown parameters m_x, σ_x can be calculated using the following formula:

$$\frac{1}{n}\sum_{i=1}^{n} e^{m(i)+0.5Covmat(i,i)} = e^{m_x+0.5\sigma_x^2}$$

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} e^{m(i)+m(j)+0.5(Covmat(i,i)+Covmat(j,j)+2Covmat(i,j))} = e^{2(m_x+\sigma_x^2)}$$

(9)

Let

$$y = \frac{1}{n}\sum_{i=1}^{n} x(i)$$

$$G = e^{y}$$

(10)

Math. expectation my and dispersion σ_y^2 y equals:

$$my = \frac{1}{n}\sum_{i=1}^{n} m(i)$$

$$\sigma_y^2 = \sum_{i=1}^{n} Covmat(i,i) + 2\sum_{i=1}^{n}\sum_{j>i}^{n} Covmat(i,j)$$

(11)

Random variables A and G are two-dimensional log-normal distribution, that is, the joint probability density equals:

$$p(A,G) = \frac{1}{AG} dnorm2(LnA, LnG, mx, my, \sigma x, \sigma y, \rho)$$

(12)

Where

$$dnorm2(LnA, LnG, mx, my, \sigma x, \sigma y, \rho) = \frac{1}{2\pi\sigma x\sigma y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\left(\frac{LnA-mx}{\sigma x}\right)^2+\left(\frac{LnG-my}{\sigma y}\right)^2-2\rho\frac{LnA-mx}{\sigma x}\frac{LnG-my}{\sigma y}\right)}$$

(13)

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In the formulas (12) and (13) there is one unknown parameter ρ. To find it we use the formula:

$$E(AG) = E(\frac{1}{n}\sum_{i=1}^{n} e^{x(i)}e^{y}) = E(e^{x+y})$$

(14)

Where

E – symbol of math. expectation.

As follows from (14)

$$P = \frac{1}{n}\sum_{i=1}^{n} e^{m(i)+my+\frac{1}{2}(Covmat(i,i)+\sigma_y^2+2Cov(x(i),y))} = e^{mx+my+\frac{1}{2}(\sigma_x^2+\sigma_y^2+2\rho\sigma_x\sigma_y)}$$

$$Cov(x(i), y) = \frac{1}{n}\sum_{k=1}^{n} Covmat(i,k)$$

$$\rho = \frac{LnP - mx - my - \frac{1}{2}(\sigma_x^2 + \sigma_y^2)}{\sigma_x\sigma_y}$$

(15)

Integral PrCall2 can be written as:

$$\Pr Call2 = e^{-rdt}\int_{K}^{\infty}\int_{0}^{K}(A-K)p(A,G)dAdG = e^{-rdt}\int_{LnK}^{\infty}\int_{-\infty}^{LnK}(e^{x}-K)dnorm2(x,y,mx,my,\sigma_x,\sigma_y,\rho)dxdy$$

(16)

For calculation (16) one is using known formulas [2]:

$$e^{x}dnorm2(x,y,mx,my,\sigma_x,\sigma_y,\rho) = Cdnorm2(x,y,Mx,My,\sigma_x,\sigma_y,\rho)$$

$$C = e^{mx+0.5\sigma_x^2}$$

$$Mx = mx + \sigma_x^2$$

$$My = my + \rho\sigma_x\sigma_y$$

(17)

Thus we obtain from formulas (16), (17):

$$\Pr Call2 = e^{-rdt}(C\int_{LnK}^{\infty}\int_{-\infty}^{LnK}(dnorm2(x,y,Mx,My,\sigma_x,\sigma_y,\rho)dxdy - K\int_{LnK}^{\infty}\int_{-\infty}^{LnK}dnorm2(x,y,mx,my,\sigma_x,\sigma_y,\rho)dxdy)$$

(18)

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Integrals in (18) are calculated using the two-dimensional cumulative normal function Φ2 (x, y, ρ):

$$\Phi 2(x,y,\rho)=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{x}\int_{-\infty}^{y} e^{-\frac{1}{2(1-\rho^2)}(u^2+v^2-2\rho uv)}dudv$$

(19)

From (19):

$$\Phi 2(-x,y,-\rho)=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{x}^{\infty}\int_{-\infty}^{y} e^{-\frac{1}{2(1-\rho^2)}(u^2+v^2-2\rho uv)}dudv$$

Finally,

$$\Pr Call2 = e^{-rdt}(C\Phi 2(-\frac{LnK-Mx}{\sigma_x},\frac{LnK-My}{\sigma_y},-\rho)-K\Phi 2(-\frac{LnK-mx}{\sigma_x},\frac{LnK-my}{\sigma_y},-\rho))$$

(20)

Reference.

[1] Michael Curran. Valuing Asian and Portfolio Options by Conditioning on the Geometric Mean Price. Portfolio Engineering. NOV-18-1993

[2] Mark Ioffe. Calculation of the integral required to calculate Asian option. Sep-18-2014